Topics in fluid mechanics

PROBLEM SHEET 0. [2018]

1. The *Boussinesq approximation* for a stratified flow assumes the density ρ is constant in the equations ($\rho = \rho_0$), except where it occurs in the gravitational buoyancy term $-\rho g \mathbf{k}$ in the Navier–Stokes momentum equation. A two-dimensional, Boussinesq fluid flow has velocity $\mathbf{u} = (u, 0, w)$, and depends only on the coordinates x and z. Show that there is a stream function ψ satisfying $u = \psi_z$, $w = -\psi_x$, and that the vorticity

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = \nabla^2 \psi \mathbf{j},$$

and thus that

$$\mathbf{u} \times \boldsymbol{\omega} = (\psi_x \nabla^2 \psi, 0, \psi_z \nabla^2 \psi),$$

and hence

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) = (\psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x)\mathbf{j}.$$

Use the vector identity $(\mathbf{u} . \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{1}{2}u^2) - \mathbf{u} \times \boldsymbol{\omega}$ to show that

$$\boldsymbol{\nabla} \times \frac{d\mathbf{u}}{dt} = \left[\nabla^2 \psi_t - \psi_x \nabla^2 \psi_z + \psi_z \nabla^2 \psi_x\right] \mathbf{j}.$$

Show also that

$$\boldsymbol{\nabla} \times \rho \mathbf{k} = -\rho_x \mathbf{j},$$

and use the Cartesian identity

$$\nabla^2 \equiv \mathrm{grad}\,\mathrm{div} - \mathrm{curl}\,\mathrm{curl}$$

to show that

$$\boldsymbol{\nabla} \times \nabla^2 \mathbf{u} = \nabla^4 \psi \, \mathbf{j}.$$

Deduce that the momentum equation can be written in the form

$$\rho_0 \left[\nabla^2 \psi_t + \psi_z \nabla^2 \psi_x - \psi_x \nabla^2 \psi_z\right] = g \rho_x + \mu \nabla^4 \psi,$$

where μ is the viscosity.

2. *The Blasius boundary layer*

Write down the dimensionless form of the Navier-Stokes equations for an incompressible viscous fluid, explaining what the Reynolds number is.

Fluid flows two-dimensionally past a flat plate $y = 0$, $x > 0$ at high Reynolds number Re, such that the dimensionless velocity (u, v) satisfies $u = v = 0$ on $y = 0$, $x > 0$, and $u \to 1$, $v \to 0$, $p \to 0$ as $y \to \infty$. Show that the outer (inviscid) flow away from $y = 0$ is $(u, v) = (1, 0)$, $p = 0$. Show that a boundary layer exists near $y = 0$, where $v \sim \delta$, $y \sim \delta$, where $\delta = Re^{-1/2}$, and show that the corresponding equations for u and (rescaled) V are

$$u_x + V_Y = 0,$$
$$uu_x + Vu_Y = u_{YY}.$$

By introducing a stream function $u = \psi_Y$, $V = -\psi_x$, deduce that

$$\psi_Y \psi_{xY} - \psi_x \psi_{YY} = \psi_{YYY}.$$

Show that a similarity solution of the form $\psi = (2x)^{1/2} f(\eta)$, $\eta = Y/(2x)^{1/2}$, exists, where f satisfies

$$f''' + ff'' = 0,$$
$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

$f(\eta)$ must be found numerically, and in common with many similarity solutions, there is a trick to do this by rescaling. Solve

$$F'''(\xi) + F(\xi)F''(\xi) = 0,$$
$$F(0) = F'(0) = 0, \quad F''(0) = 1;$$

(this can be done easily as an initial value problem, providing (as is the case) F' cannot blow up at ∞)). Put $f(\eta) = bF(a\eta)$, and show that the required solution is obtained by taking

$$a = b = \sqrt{\frac{1}{F'(\infty)}}.$$

Sketch the graph of $f'(\eta)$. What does it represent?

3. [*This is difficult.*] A viscous, incompressible fluid of density ρ and mean depth d flows slowly down a rough surface inclined at an angle α to the horizontal. The flow is two-dimensional, and is driven by the downslope gravitational acceleration. If (x, z) are cartesian coordinates, with x pointing downslope, write down the equations of Stokes flow for the pressure p and stream function ψ, in which the inertial terms are neglected.

   If the boundary conditions are of no slip at the base $z = b(x)$ and no normal stress at the top surface $z = s(x, t)$, $\sigma_{nn} + p_a = \sigma_{nt} = 0$, where p_a is atmospheric pressure, show that we can take

$$\psi = \psi_z = 0 \quad \text{at} \quad z = b,$$

and

$$(p - p_a)(1 + s_x^2) + 2s_x\tau_{13} + (1 - s_x^2)\tau_{11} = 0,$$

$$(1 - s_x^2)\tau_{13} - 2s_x\tau_{11} = 0 \quad \text{at} \quad z = s,$$

where you should define the normal deviatoric stress τ_{11} and the shear stress τ_{13} in terms of derivatives of ψ.

Write down the kinematic condition for the free surface, and show that it can be written in the form

$$s_t + \frac{\partial\psi[x, s(x,t)]}{\partial x} = 0.$$

By choosing suitable scales for the reduced pressure $p - p_a - \rho g(s - z)\cos\alpha$, stream function ψ, lengths x, z, s, b and time t, show that the equations can be written in the dimensionless form (the variables are all now dimensionless, in particular p is the dimensionless *reduced* pressure)

$$s_x \cot\alpha + p_x = \nabla^2\psi_z + 1,$$

$$p_z = -\nabla^2\psi_x,$$

and write down the corresponding dimensionless boundary conditions. Show that the velocity scale U is given by

$$U = \frac{\rho g d^2 \sin\alpha}{\eta},$$

where η is the viscosity.

Now assume the flow is steady. Show in this case that ψ is constant on $z = s$. For the particular case $b = 0$, find an exact steady solution in which $s = 1$, and show that in this case $\psi = \frac{1}{3}$ at $z = 1$.

Next, suppose that b and thus $s - 1$ are small, and that the downslope volume flux is prescribed, $\psi = \frac{1}{3}$ at $z = s$. By writing

$$\psi = \tfrac{1}{2}z^2 - \tfrac{1}{6}z^3 + \Psi, \quad s = 1 + \sigma, \quad P = p + \sigma\cot\alpha,$$

show that

$$P_x = \nabla^2\Psi_z, \quad P_z = -\nabla^2\Psi_x,$$

and show that linearised boundary conditions can be taken to be

$$\Psi_{zz} - \Psi_{xx} = \sigma, \quad \Psi + \tfrac{1}{2}\sigma = 0, \quad P + 2\Psi_{zx} = \sigma\cot\alpha \quad \text{at} \quad z = 1,$$

$$\Psi = 0, \quad \Psi_z = -b \quad \text{at} \quad z = 0.$$

Explain how the solution of this problem enables the determination of the surface perturbation σ.

[*For the foolhardy: to solve the problem in terms of Fourier transforms, write* $\Psi = f(z)e^{ikx}$, $P = g(z)e^{ikx}$, $b = Be^{ikx}$, $\sigma = \Sigma e^{ikx}$, *show that* $g = (f''' - k^2 f')/ik$, *that* $f = az\cosh kz + (bz + c)\sinh kz$, *and that* $f'(0) = -B$, *and also* $f''(1) + (k^2+2)f(1) = 0$, $\Sigma = -2f(1)$, *and* $f'''(1) - 3k^2 f'(1) + 2ikf(1)\cot\alpha = 0$. *Hence deduce that* $\Sigma = KB$, *where*

$$K = \frac{2\cosh k}{1 + k^2 + \cosh^2 k - \dfrac{i\cot\alpha}{k^2}(\sinh k\cosh k - k)}.$$

Note that $K(0) = 1$, *as it must (why?)*]

4. Write down the equations and boundary conditions suitable to describe the motion of a layer of incompressible, inviscid fluid of mean depth h subject to a gravity force in the downwards z direction. Explain what it means for the flow to be irrotational, and in this case show that there is a velocity potential ϕ, and that (if the bed of the fluid is at $z = -h$ and the surface is at $z = \eta$)

$$\begin{aligned} \nabla^2\phi &= 0, \\ \phi_z = 0 \quad &\text{at} \quad z = -h, \\ \phi_z = \eta_t + \boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\eta \quad &\text{at} \quad z = \eta. \end{aligned}$$

Show that the quantity $\dfrac{p - p_a}{\rho} + \phi_t + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + gz$ is constant in the fluid (p_a is atmospheric pressure), and deduce a second boundary condition for the flow if $p = p_a$ at $z = \eta$.

A stream of depth h flows at constant speed U in the x direction and is uniform in the far field (thus $\phi = Ux$, $\eta = 0$). Show that these far field conditions define a uniformly valid solution for ϕ and η.

Now consider a small disturbance to the flow, so that η and $\Phi = \phi - Ux$ are small. By linearising about the uniform state, write down a linear set of differential equations and boundary conditions for the perturbed velocity potential Φ and η, and by solving this, derive the dispersion relation relating wave speed c to wave number k in the form

$$c = U \pm \sqrt{\frac{g}{k}\tanh kh}.$$

Interpret this result physically.

5. Show that the equation describing conservation of mass of a shallow, incompressible, inviscid flow in $0 < z < h$ is

$$h_t + \boldsymbol{\nabla}.\left[\int_0^h \mathbf{u}\,dz\right] = 0,$$

where $\mathbf{u} = (u, v, 0)$ is the horizontal velocity vector.

Show further that the horizontal component of momentum conservation,

$$\mathbf{u}_t + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} + w\mathbf{u}_z = -\frac{1}{\rho}\boldsymbol{\nabla}p,$$

where w is the vertical component of velocity, and $\boldsymbol{\nabla}$ is the horizontal gradient vector, together with a hydrostatic balance

$$p = p_a + \rho g(h - z),$$

lead, when integrated from $z = 0$ to $z = h$ using the kinematic condition

$$w = h_t + \mathbf{u}.\boldsymbol{\nabla}h \quad \text{at} \quad z = h,$$

to the integrated form

$$\frac{\partial}{\partial t}\int_0^h \mathbf{u}\,dz + \boldsymbol{\nabla}.\left[\int_0^h (\mathbf{u}\mathbf{u})\,dz\right] + gh\boldsymbol{\nabla}h = \mathbf{0}.$$

Deduce the (two-dimensional) form of the shallow water equations if it is assumed that $\mathbf{u}$ is independent of z.

[*The dyadic* $\mathbf{uu}$ *is the tensor with components* $u_i u_j$*, and the divergence of a tensor* $\boldsymbol{\sigma}$ *is the vector with i-th component* $\partial\sigma_{ij}/\partial x_j$*, where summation over* j *is understood.*]

6. A train of (one-dimensional) ocean waves approaches the shore at $x = 0$ from $x = +\infty$ over a sloping base at $z = -b(x)$; the undisturbed sea surface is at $z = 0$, and the disturbed surface is $z = \eta(x, t)$, so that the water depth is $h = \eta + b$.

   Show that the no flow through condition at $z = -b$ takes the form

$$w = -ub'.$$

   Derive the shallow water equations from first principles, and show that they take the form

$$h_t + (hu)_x = 0,$$

$$u_t + uu_x + g\eta_x = 0.$$

   Hence show that if $m = gb'(x)$ is constant, the Riemann invariants are $u \pm 2c - mt$ on $\dot{x} = u \pm c$, where $c = \sqrt{gh}$.

   Suppose that at $t = 0$, $u = u_0(x)$ and $c = c_0(x) = K - \frac{1}{2}u_0(x)$. Show that $u + 2c - mt = 2K$ everywhere, and deduce that on the negative characteristics through $x = \xi$, $t = 0$, ,

$$u = u_0(\xi) + mt \quad \text{and} \quad x = \xi + \left[\tfrac{3}{2}u_0(\xi) - K\right]t + \tfrac{1}{2}mt^2,$$

and deduce that

$$u = mt + u_0 \left[x - \tfrac{3}{2}ut + Kt + mt^2 \right].$$

Hence show that waves will break (i. e., a shock forms) if $c_0'(\xi) > 0$ anywhere.

Do these initial conditions make any physical sense?